On the potential theory of subordinate killed processes

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Joint work with P. Kim (SNU) and R. Song (UIUC)

Killed stable process

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For $D \subset \mathbb{R}^d$ open, let $\tau_D := \inf\{t > 0 : Z_t \notin D\}$, $Z_t^D := Z_t$ if $t < \tau_D$, ∂ (cemetary) otherwise, $Q_t^D f(x) := \mathbb{E}_x f(Z_t^D) = \mathbb{E}_x(f(Z_t), t < \tau_D)$ the corresponding semigroup.

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$$\mathcal{L}_1 f := \lim_{t\to 0} \frac{Q_t^D f - f}{t}$$

a possible definition of fractional Laplacian in D; usually called *fractional Laplacian in D with zero exterior condition*. Notation: $-(-\Delta)^\alpha_{|D}$.

KSBM and SKBM

Let $W = (W_t, \mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$, $S = (S_t)_{t \geq 0}$ an independent α-stable subordinator. Then W_{S_t} is a subordinate Brownian motion and $(Z_t) \overset{d}{=} (W_{S_t})$.

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W^D Brownian motion killed upon exiting D, $Y_t^D := W^D_{S_t}$ is a subordinate killed Brownian motion (SKBM). If $(P_t^D)_{t\geq 0}$ is the semigroup of W^D, then the infinitesimal generator of Y^D is

$$\mathcal{L}_0 f = -(-\Delta_{|D})^{\alpha} f := \frac{1}{|\Gamma(-\alpha)|}\int_0^\infty (P_t^D f - f)t^{-\alpha-1}dt$$

Another possible definition of a fractional Laplacian in D: *the fractional power of Dirichlet Laplacian*.

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Another possible definition of a fractional Laplacian in D: *the fractional power of Dirichlet Laplacian*. $\mathcal{L}_0\neq\mathcal{L}_1$

If $(\widetilde{Q}_t^D)_{t\geq 0}$ is the semigroup of Y^D, then $\widetilde{Q}_t^D f \leq Q_t^D f$, $f \geq 0$. Y^D is a "smaller" process then Z^D.

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Let S be a β-stable subordinator, $\beta \in (0,1]$, T a γ-stable subordinator, $\gamma \in (0,1)$, so that $\beta\gamma = \alpha$. Let $Z_t = W_{S_t}$ be a SBM, Z^D the KSBM, and $Y_t^D = Z_{T_t}^D$ the subordinate killed Z.

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$$\mathcal{L} = -((-\Delta)^{\beta}_{|D})^{\gamma} .$$

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Goal: study potential theory of operators like $\mathcal{L}$ (i.e. potential theory of Y^D) and see how it depends on β and γ.

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Various processes:

$Z_t := W_{S_t}$ subordinate Brownian motion; BM or 2β-stable

$X_t := Z_{T_t} = W_{(S\circ T)_t}$ subordinate Z, twice SBM: 2α-stable

Z^D killed Z, KSBM

X^D killed X, KSBM

$Y_t^D := Z_{T_t}^D$ subordinate Z^D, subordinate KSBM

κ-fat and $C^{1,1}$-open sets

Let $0 < \kappa < 1$. An open set $D \subset \mathbb{R}^d$ is said to be κ-fat if there is $R_1 > 0$ such that for all $x \in \overline{D}$ and all $r \in (0, R_1]$, there is a ball $B(A_r(x), \kappa r) \subset D \cap B(x, r)$. The pair (R_1, κ) is called the characteristics of the κ-fat open set D.

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Let $U \subset \mathbb{R}^d$ be an open set and let $Q \in \partial U$. We say that U is $C^{1,1}$ near Q if there exist a localization radius $R > 0$, a $C^{1,1}$-function $\varphi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi_Q(0) = 0$, $\nabla\varphi_Q(0) = (0, \dots, 0)$, $\|\nabla\varphi_Q\|_\infty \le \Lambda$, $|\nabla\varphi_Q(z) - \nabla\varphi_Q(w)| \le \Lambda|z - w|$, and an orthonormal coordinate system CS_Q with its origin at Q such that $B(Q, R) \cap U = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS_Q : \ y_d > \varphi_Q(\widetilde{y})\}$ where $\widetilde{y} := (y_1, \dots, y_{d-1})$. The pair (R, Λ) is called the $C^{1,1}$ characteristics of U near Q. An open set $U \subset \mathbb{R}^d$ is said to be a (uniform) $C^{1,1}$ open set with characteristics (R, Λ) if it is $C^{1,1}$ with characteristics (R, Λ) near every boundary point $Q \in \partial U$.

Transition densities of Z: $p(t,x,y) = p(t,|x-y|)$ where

$$p(t,r) = (4\pi t)^{-d/2} e^{-r^2/4t}\,, \quad S_t = t\,, \quad (Z = W)$$

$$p(t,r) = \int_0^\infty (4\pi s)^{-d/2} e^{-r^2/4s} \mathbb{P}(S_t \in ds)\,, \quad \text{otherwise.}$$

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Transition densities of Z^D:

$$p^D(t,x,y)=p(t,x,y)-\mathbb{E}_x[p(t-\tau_D,Z_{\tau_D},y),\tau_D<t]\,,\quad t>0,\ x,y\in D\,.$$

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Transition densities of Z^D:

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Sharp two-sided estimates: (i) $\phi(\lambda) = \lambda$ ($\beta = 1$); for $t \le T$, $x, y \in D$,

$$p^D(t,x,y) \asymp \mathbb{P}_x(t < \tau_D^W)\, \mathbb{P}_y(t < \tau_D^W) t^{-d/2} e^{-\frac{c|x-y|^2}{t}},$$
$$p^D(t,x,y) \asymp \left(\frac{\delta_D(x)}{t^{1/2}} \wedge 1\right)\left(\frac{\delta_D(y)}{t^{1/2}} \wedge 1\right) t^{-d/2} e^{-\frac{c|x-y|^2}{t}}$$

Varopoulos 2003 (D Lipschitz), Zhang 2003, Song 2004 (D $C^{1,1}$)

(ii) $t \le T$, $x, y \in D$, D is κ-fat open set,

$$p^D(t,x,y) \asymp \mathbb{P}_x(t < \tau_t^Z)\mathbb{P}_y(t < \tau_D^Z)\left(t^{-d/2\beta} \wedge \frac{t}{|x-y|^{d+2\beta}}\right).$$

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D is $C^{1,1}$ open set, $x, y \in D$,

$$\begin{aligned} p^D(t,x,y) &\asymp \left(\frac{\delta_D(x)^\beta}{t^{1/2}} \wedge 1\right)\left(\frac{\delta_D(y)^\beta}{t^{1/2}} \wedge 1\right) \\ &\quad \times \left(t^{-d/2\beta} \wedge \frac{t}{|x-y|^{d+2\beta}}\right), \quad t \le T, \\ p^D(t,x,y) &\asymp e^{-\lambda_1 t}\delta_D(x)^\beta \delta_D(y)^\beta, \quad t \ge T. \end{aligned}$$

Chen, Kim, Song 2010

Green function and jumping kernel

Green functions of Y^D and X: For $x, y \in D$,

$$G^{Y^D}(x,y) = \int_0^\infty p^D(t,x,y)v(t)\,dt \le \int_0^\infty p(t,x,y)v(t)\,dt = G^X(x,y)\,.$$

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Jumping kernels of of Y^D and X: For $x, y \in D$,

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Here $\nu(t) = t^{-\gamma-1}$ is the Lévy density of (T_t).

Harmonic functions

$(X_t, \mathbb{P}_x)$ a strong Markov process in a metric space $\mathfrak{X}$. A non-negative $u : \mathfrak{X} \to [0, \infty)$ is harmonic in an open $U \subset \mathfrak{X}$ (wrt the process X) if

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A non-negative $u : \mathfrak{X} \to [0, \infty)$ is regular harmonic in an open $U \subset \mathfrak{X}$ if

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Harnack inequality: There exists $C > 0$ such that for every $x_0 \in \mathfrak{X}$, every $r \in (0, 1]$ and every function $u : \mathfrak{X} \to [0, \infty)$ which is harmonic in $B(x_0, r)$ with respect to X, it holds that

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C does *not* depend on $r \in (0, 1]$ - scale invariant HI.

Boundary Harnack principle

BHP for X: Let $U \subset \mathfrak{X} \subset \mathbb{R}^d$. There exists $C > 0$ such that for every $r \in (0,1]$, every $Q \in \partial U$, any nonegative f and g on $\mathbb{R}^d$ which are regular harmonic in $U \cap B(Q,r)$ and vanish on $U^c \cap B(Q,r)$,

$$\frac{f(x)}{g(x)} \leq C \frac{f(y)}{g(y)}, \quad x, y \in U \cap B(Q, r/2).$$

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BHP with explicit decay rate: When U is $C^{1,1}$, replace g with an explicit function of $\delta_U(\cdot)$.

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BHP with explicit decay rate: When U is $C^{1,1}$, replace g with an explicit function of $\delta_U(\cdot)$.
For example, if X is Brownian motion, then $\frac{f(x)}{\delta_U(x)} \leq C \frac{f(y)}{\delta_U(y)}$. If X is isotropic 2α-stable, then $\frac{f(x)}{\delta_U(x)^\alpha} \leq C \frac{f(y)}{\delta_U(y)^\alpha}$.

Harnack inequalities

Theorem: Suppose that $D \subset \mathbb{R}^d$ is a bounded κ-fat (Lipschitz when $\beta = 1$) open set. There exists a constant $C > 0$ such that for any $r \in (0, 1]$ and $B(x_0, r) \subset D$ and any Borel function f which is non-negative in D and harmonic in $B(x_0, r)$ with respect to Y^D, we have

$$f(x) \leq Cf(y), \qquad \text{for all } x, y \in B(x_0, r/2).$$

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Theorem: Suppose that $D \subset \mathbb{R}^d$ is a bounded κ-fat open set, $\beta \in (0,1)$. There exists a constant $C = C(\beta, \gamma, \mathrm{diam}(D)) > 1$ such that the following is true: If $L > 0$ and $x_1, x_2 \in D$ and $r \in (0,1)$ are such that $|x_1 - x_2| < Lr$ and $B(x_1, r) \cup B(x_2, r) \subset D$, then for any Borel function f which is non-negative in D and harmonic in $B(x_1, r) \cup B(x_2, r)$ with respect to Y^D, we have

$$C^{-1}(L \vee 1)^{-d-\beta} f(x_2) \le f(x_1) \le C(L \vee 1)^{d+\beta} f(x_2).$$

One of the main ingredients of the proof is the following comparison of jumping kernel.

Lemma: Suppose that $D \subset \mathbb{R}^d$ is a bounded κ-fat (Lipschitz when $\beta = 1$) open set. For every $\varepsilon_0 \in (0, 1]$, there exists a constant $C \geq 1$ such that for all $x_0 \in D$ and all $r \leq 1$ satisfying $B(x_0, (1 + \varepsilon_0)r) \subset D$, it holds that

$$J^{Y^D}(z, x_1) \leq C J^{Y^D}(z, x_2), \quad x_1, x_2 \in B(x_0, r), \ \ z \in D \setminus B(x_0, (1 + \varepsilon_0)r).$$

Boundary Harnack principle in the interior of D

Theorem: Let $D \subset \mathbb{R}^d$, $d \geq 3$, be a bounded κ-fat (Lipschitz when $\beta = 1$) open set. There exists a constant $b = b(\beta, \gamma, d) > 2$ such that, for every open set $E \subset D$ and every $Q \in \partial E \cap D$ such that E is $C^{1,1}$ near Q with characteristics $(\delta_D(Q) \wedge 1, \Lambda)$, the following holds: There exists a constant $C = C(\delta_D(Q) \wedge 1, \Lambda, \psi, \phi, d) > 0$ such that for every $r \leq (\delta_D(Q) \wedge 1)/(b+2)$ and every non-negative function f on D which is regular harmonic in $E \cap B(Q, r)$ with respect to Y^D and vanishes on $E^c \cap B(Q, r)$, we have for $x, y \in E \cap B(Q, 2^{-6}\kappa_0^4 r)$,

$$\frac{f(x)}{\delta_E(x)^\alpha} \leq C \frac{f(y)}{\delta_E(y)^\alpha},$$

where $\kappa_0 = (1 + (1+\Lambda)^2)^{-1/2}$.

Green function estimates in $C^{1,1}$-open set

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$$G^{Y^D}(x,y) \asymp^C \left(\frac{\delta_D(x)^\beta}{|x-y|^\beta} \wedge 1\right)\left(\frac{\delta_D(y)^\beta}{|x-y|^\beta} \wedge 1\right) g(|x-y|).$$

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The boundary behavior $\delta_D(x)^\beta$ completely determined by ϕ (that is, β); same as the one of Z^D.

Outline of the proof: With $T = 2\mathrm{diam}(D)^{2\beta}$, write

$$G^{Y^D}(x,y) = \int_0^\infty p^D(t,x,y)v(t)dt = \int_0^{|x-y|^{2\beta}} + \int_{|x-y|^{2\beta}}^T + \int_T^\infty,$$

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use sharp estimates of $p^D(t,x,y)$ and $v(t) = t^{\gamma-1}$.
For the upper bound estimate all three integrals, for the lower only the first which is the dominating term.

Exit time estimates in $C^{1,1}$ domains

Recall that $\mathbb{E}_x \tau_D = \int_D G^{Y^D}(x,y)dy$ (in fact, $\tau_D = \zeta$ - the lifetime of Y^D).

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It holds that

$$\mathbb{E}_x\tau_D \asymp \begin{cases} \delta_D(x)^\beta, & \gamma > 1/2, \\ \delta_D(x)^\beta \log(1/\delta_D(x)), & \gamma = 1/2, \\ \delta_D(x)^{2\gamma\beta}, & \gamma < 1/2\,. \end{cases}$$

Jumping kernel estimates in $C^{1,1}$-open sets

Let

$$j(r) := \frac{1}{r^{d+2\alpha}}, \quad r > 0.$$

Note that $j(|x-y|)$ is the Lévy density estimate for X.

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Theorem: Suppose $\phi(\lambda) = \lambda$ (that is $\beta = 1$). There exists a constant $C \geq 1$ such that for all $x, y \in D$,

$$J^{Y^D}(x,y) \asymp^C \left(\frac{\delta_D(x)}{|x-y|} \wedge 1 \right) \left(\frac{\delta_D(y)}{|x-y|} \wedge 1 \right) j(|x-y|).$$

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Boundary behavior same as for the Green function. The proof similar, uses $J^{Y^D}(x,y) = \int_0^\infty p^D(t,x,y) t^{-\gamma-1} dt$,

Jumping kernel estimates in $C^{1,1}$-open sets (cont.)

Case $\phi(\lambda) \neq \lambda$ (that is $\beta \in (0, 1)$).

Jumping kernel estimates in $C^{1,1}$-open sets (cont.)

Case $\phi(\lambda) \neq \lambda$ (that is $\beta \in (0,1)$).

$$J^{Y^D}(x,y) \asymp \begin{cases} \left(\frac{\delta_D(x)\wedge\delta_D(y))^{2\beta}}{|x-y|^{2\beta}} \wedge 1\right)^{1-\gamma} \frac{1}{|x-y|^{d+2\gamma\beta}}, & \gamma > 1/2, \\ \left(\frac{\delta_D(x)\wedge\delta_D(y))^{2\beta}}{|x-y|^{2\beta}} \wedge 1\right)^{1/2}, & \\ \times \log\left(1 + \frac{(\delta_D(x)\vee\delta_D(y))^{2\beta}\wedge|x-y|^{2\beta}}{(\delta_D(x)\wedge\delta_D(y))^{2\beta}\wedge|x-y|^{2\beta}}\right) \frac{1}{|x-y|^{d+\beta}}, & \gamma = 1/2, \\ \left(\frac{\delta_D(x)\wedge\delta_D(y))^{2\beta}}{|x-y|^{2\beta}} \wedge 1\right)^{1/2} & \\ \times \left(\frac{(\delta_D(x)\vee\delta_D(y))^{2\beta}}{|x-y|^{2\beta}} \wedge 1\right) \frac{1}{|x-y|^{d+2\gamma\beta}}, & \gamma < 1/2. \end{cases}$$

Example

Let $\psi(\lambda) = \lambda^\gamma$ and assume that $\beta \in (0,1)$. Fix $y \in D$. As $\delta_D(x) \to 0$, we have

$$J^{Y^D}(x,y) \asymp^c \begin{cases} \delta_D(x)^\beta, & 0 < \gamma < 1/2, \\ \delta_D(x)^\beta \log(1/\delta_D(x)), & \gamma = 1/2, \\ \delta_D(x)^\beta \, \delta_D(x)^{\beta - 2\gamma\beta} = \delta_D(x)^{2\beta(1-\gamma)}, & 1/2 < \gamma < 1. \end{cases}$$

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In case $\phi(\lambda)=\lambda$, in all three cases

$$J^{Y^D}(x,y)\asymp^c \delta_D(x)\,.$$

The proof of the theorem uses that

$$J^{Y^D}(x,y) = \int_0^\infty p^D(t,x,y)\nu(t)dt\,,$$

estimates of $p^D(t,x,y)$ and $\nu(t) = t^{-\gamma-1}$. The integral is split into three parts:

$$\int_0^{|x-y|^{2\beta}} + \int_{|x-y|^{2\beta}}^T + \int_T^\infty\,.$$

The last two integrals are estimated from above in a rather straightforward way, but the first one is quite delicate. Estimates used for the Green function do not work, because some of the integrals diverge.

Carleson estimate

Theorem (Carleson estimate): There exists a constant $C = C(R, \Lambda) > 0$ such that for every $Q \in \partial D$, $0 < r < R/2$, and every non-negative function f in D that is harmonic in $D \cap B(Q, r)$ with respect to Y^D and vanishes continuously on $\partial D \cap B(Q, r)$, we have

$$f(x) \le Cf(x_0) \qquad \text{for } x \in D \cap B(Q, r/2),$$

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In case $\phi(\lambda) = \lambda$ the Carleson estimate is proved for $C^{1,1}$-open sets by using the jumping kernel estimates.
In case $\phi(\lambda) \neq \lambda$, the Carleson estimate is proved when D is κ-fat and satisfies the local exterior volume condition. In this case, the proof does not use the explicit estimates for J^{Y^D}.

Carleson estimate, cont.

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Carleson estimate, cont.

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(2) a parabolic Carleson-type estimate for Z^D: For any $T>0$ and $c_0 \in (0,1)$, there exists $C = C(R_1, \kappa, c_0, T) \geq 1$ such that for all $t \in (0, T]$, $r \leq R_1/2$, $Q \in \partial D$ and $x, x_0 \in D \cap B(Q,r)$ with $\delta_D(x_0) \geq c_0 r$,

$$\mathbb{P}_x(\tau_D^Z > t) \leq C\mathbb{P}_{x_0}(\tau_D^Z > t).$$

Theorem: Assume that $\beta = 1$, or $\beta \in (0,1)$ and $\gamma > 1/2$. Let $D \subset \mathbb{R}^d$ be a bounded $C^{1,1}$ open set with $C^{1,1}$ characteristics (R, Λ). There exists a constant $C = C(d, \Lambda, R, \beta, \gamma) > 0$ such that for any $r \in (0, R]$, $Q \in \partial D$, and any non-negative function f in D which is harmonic in $D \cap B(Q, r)$ with respect to Y^D and vanishes continuously on $\partial D \cap B(Q, r)$, we have

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As a consequence, the rate of decay of harmonic functions with respect to Y^D is given by $\delta_D(x)^\beta$, depends on β only, and is equal to the decay of harmonic functions with respect to Z.

Failure of BHP for $\gamma \le 1/2$

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$$\frac{f(x)}{f(y)} \le c\,\frac{g(x)}{g(y)} \qquad \text{for all } x, y \in D \cap B(Q, r/2).$$

By taking $g = G^{Y^D}(\cdot, w)$, $w \notin D \cap B(Q,r)$, and by using the Green function estimates, we get that for any $r \in (0, \widehat{R}]$ there is $C = C(r) > 0$ such that for every $Q \in \partial D$ and any non-negative function f as above

$$\frac{f(x)}{f(y)} \le C\,\frac{\delta_D(x)^\beta}{\delta_D(y)^\beta}, \quad \text{for all } x, y \in D \cap B(Q, r/2).$$

Since D satisfies the interior ball condition, there exist r_0 small enough and $x^{(1)} \in B(Q,R) \cap D$ with $\delta_D(x^{(1)}) = r_0$ such that $\delta_D(x^{(s)}) = |x^{(s)} - Q|$ for all $s \le 1$ where $x^{(s)} = Q + s(x^{(1)} - Q)$.

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For $n \in \mathbb{N}$ large enough so that $B(z_0, 1/n)$ does not intersect $B(Q, 2r_0)$, we define (for $\gamma < 1/2$; the case $\gamma = 1/2$ is similar),

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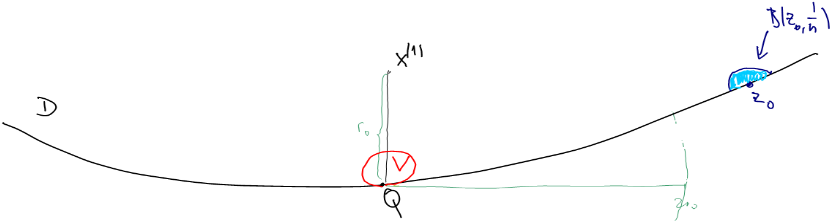
Here $V = V_Q(2^{-2}\kappa_0 r_0)$ is a small $C^{1,1}$ set near Q.



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Lemma: There exists $C > 0$ such that

$$\liminf_{n\to\infty} g_n(x) \geq C\delta_D(x)^\beta \log(r_0/\delta_D(x))$$

for all $x = x^{(s)} = (\widetilde{0}, s)$ in CS with s small enough.

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One can also show that for large n and all $y \in D \cap B(Q, 2^{-7}\kappa_0 r_0)$,

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so that g_n are harmonic in $D \cap B(Q, 2^{-7}\kappa_0 r_0)$ and vanish continuously on $\partial D \cap B(Q, 2^{-7}\kappa_0 r_0)$.

Therefore (by the assumption that BHP holds)

$$\frac{g_n(y)}{g_n(w)} \le C \frac{\delta_D(y)^\beta}{\delta_D(w)^\beta} \quad \text{for all } y \in D \cap B(Q, 2^{-8}\kappa_0 r_0)$$

where $w = (\widetilde{0}, 2^{-9}\kappa_0 r_0)$ and $C = C(2^{-7}\kappa_0 r_0)$.

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By the upper estimate of g_n it follows that

$$\limsup_{n\to\infty} g_n(y) \le C \limsup_{n\to\infty} g_n(w)\frac{\delta_D(y)^\beta}{\delta_D(w)^\beta} \le c\delta_D(y)^\beta.$$

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By Lemma, for $y = x = x^{(s)}$, the left-hand side above is bounded from below by $C\delta_D(x)^\beta \log(r_0/\delta_D(x))$, yielding

$$\log(r_0/\delta_D(x)) \le \frac{c}{C},$$

which is a contradiction.